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New Class of Dual Vertices

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ABSTRACT

A new $SU(1,1)$ coupling scheme, based on the discrete class

representations, is proposed to construct factorizable dual amplitudes.

We introduce dual vertices for off-shell particles of even spin and

discuss other applications of the method. Our analysis suggests the

non-existence of a dual vector current within the context of the conven-

tional Veneziano model.

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Recently, general criteria[1] for the construction of dual amplitudes

have been formulated based on a continuous class representation of

$SU(1,1)$[2]. Unfortunately, the coupling scheme between the $SU(1,1)$-

covariant vertices broke down when the external particles were taken off

their mass-shells, thus preventing the construction of dual current

amplitudes. In this letter we present similar general criteria for the

construction of dual amplitudes based on the discrete class representations

of $SU(1,1)$. In this way, we obtain amplitudes of a kind that have recently

arisen in the generalization of the Virasoro amplitude[3] as well as in the

context of dual treatments of the pomeron[4] and of scalar currents[5].

Again, we point out that these rules are more general than any particular

dual model, thereby supporting the fundamental connection between $SU(1,1)$

and the duality property.

We consider amplitudes of the form

$$A_N(k_1, \ldots k_N) = \langle 0 | \prod_{i=1}^{N} V(k_i) | 0 \rangle \, \delta^4 \left(\sum_{i=1}^{N} k_i \right) \tag{1}$$

where $V(k_i)$ represents the absorption of a particle of momentum k_i, and

is built in the following way. Take operator functions of k and z, $G(k,z)$

and $H(k,z)$, each of which transforms as a spin J representation of the

discrete class of $SU(1,1)$.

That is, under a finite transformation (see Eq. (A-20) of Ref. 2)

$$e^{i \vec{\zeta} \cdot L} G(k,z) e^{-i \vec{\zeta} \cdot L} = (\alpha^* - \beta z)^{2J} G(k,z') \qquad (2)$$

where

$$z' = \frac{\alpha z - \beta^*}{\alpha^* - \beta^* z} \qquad (3)$$

and the $L_i (i=1,2,3)$ are the generators of the algebra. Identical equations hold for $H(k,z)$. Then $V(k)$ is given by the invariant inner product over the group space (see Eq. (A-19) of Ref. 2)

$$V(k) = \int d^2 z \ (1-|z|^2)^{-2J-2} \ G^\dagger(k,z) H(k,z) \qquad (4)$$

where the integration extends over the unit circle. Clearly $A_N(k_1, \ldots, k_N)$ is an SU(1,1) invariant. Strictly speaking, it is not necessary for G and H to be separately covariant, for any function K(z) satisfying

$$e^{i \vec{\zeta} \cdot L} K(z) e^{-i \vec{\zeta} \cdot L} = |\alpha^* - \beta z|^{4J} K(z') \qquad (5)$$

where z is an arbitrary point inside the unit circle will suffice in place of $G^\dagger H$.

As an example of the above compling scheme, we might take

$$H(k, z) = e^{i \frac{k}{\sqrt{2}} \cdot \tilde{F}_b(z)} \, e^{i \frac{k}{\sqrt{2}} \cdot F_a(z)}$$

(6)

where

$$F_{a\rho}(z) = \sum_{m=0}^{\infty} \left[\frac{(m-1+\epsilon)!}{m!} \right]^{1/2} a_\rho^m \, z^{n+\epsilon/2}$$

(7a)

$$\tilde{F}_{b\rho}(z) = \sum_{m=0}^{\infty} \left[\frac{(m-1+\epsilon)!}{m!} \right]^{1/2} b_\rho^{m\dagger} \, z^{-m-\epsilon/2}$$

(7b)

with[6]

$$[a_\rho^n, a_\sigma^{m\dagger}] = [b_\rho^n, b_\sigma^{m\dagger}] = g_{\rho\sigma} \, \delta_{mn}$$

(8a)

$$[a, b] = [a, b^\dagger] = 0$$

(8b)

and ϵ is an infinitesimal taken to zero at the end of all calculations.
In addition we choose

$$G(k, z) = H(-k, z)$$

(9)

It can be checked that both H and G satisfy the requirement[*] of

Eq. 2 with J=0, where the SU (1,1) generators are now the sum

of the generators made out of a and b operators.

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[*] Actually H and G are covariant only up to terms of order ϵ
which become harmless in the combination $G^\dagger H$.

The SU (1,1) invariant vertex is then given by

$$V(k) = \int d^2z \, (1-|z|^2)^{-2} \, e^{\frac{i k}{\sqrt{2}} \cdot \tilde{F}_a(\bar{z})} \, e^{\frac{i k}{\sqrt{2}} \cdot \check{F}_b(z)} \, e^{\frac{i k}{\sqrt{2}} \cdot \check{F}_b(z)} \, e^{\frac{i k}{\sqrt{2}} \cdot F_a(z)}$$

(10a)

$$= \int d^2z \, (1-|z|^2)^{\frac{k^2}{2}-2} \, |z|^{-k^2/2} \, e^{\frac{i k}{\sqrt{2}} \cdot [\tilde{F}_a(\bar{z}) + \check{F}_b(z)]} \, e^{\frac{i k}{\sqrt{2}} \cdot [F_a(z) + F_b(\bar{z})]}$$

(10b)

where we have used

$$F^\dagger(z) = \tilde{F}(\bar{z})$$

(11)

with $\bar{z}$ being the electrostatic reflection of z with respect to the unit circle

$$\bar{z} = z^{*-1}$$

(12)

We note that this vertex retains its SU (1,1) invariance when k^2 is taken of mass shell. The N point amplitude corresponding to this vertex is obtained from Eq. 1 and is given by

$$A_N(k_1, \ldots k_N) = \left\{ \prod_{i=1}^{N} \frac{d^2 z_i}{(1-|z_i|^2)^2} \right\} \prod_{i,j=1}^{N} |z_i - \bar{z}_j|^{k_i \cdot k_j/2}$$

(13)

An expression of this form was proposed by Rebbi and Drummond[4] in connection with scalar currents. As shown by these authors, the amplitude has poles at $\frac{1}{2} k_i^2 = 1, 0, -1, -2, \ldots$ which can be directly seen by expanding the integrand of V(k) around the unit circle:

$$V(k) = \int_0^{2\pi} d\theta \sum_{n=0}^{\infty} \frac{R_N(k, \theta)}{\alpha(-k^2) - N}$$

(14)

with $\alpha_0 = 1$. The residue of the 1st pole is the on-shell Veneziano

vertex written in terms of the operators

$$C_\rho^{n\dagger} = \frac{i}{\sqrt{2}} \left(a_\rho^{n\dagger} + b_\rho^{n\dagger} \right)$$

(15)

i. e.

$$R_o(\theta, k) = e^{ik \cdot \breve{F}_c(e^{i\theta})} \, e^{ik \cdot F_c(e^{i\theta})}$$

(16)

The residue of the second pole, however, is antisymmetric under

interchange of a and b operators and thus decouples from an arbitrary

number of on-shell mesons represented by the vertex (16).

The building of off-shell vertices for even spin particles is

relatively straightforward, using for H and G on-shell vertices

previously derived.[1,7] As an example, for spin two, we start from

$$H_\mu(k, z) = z^{-1} \left[P_{\mu a}(z) + \breve{P}_{\mu b}(z) \right] e^{i\frac{k}{\sqrt{2}} \cdot [\breve{F}_b(z) + F_a(z)]}$$

(17)

where

$$P_{\mu a}(z) = -i z \frac{d}{dz} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) F_{\nu a}(z)$$

(18a)

$$\breve{P}_{\mu b}(z) = -i z \frac{d}{dz} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \breve{F}_{\nu b}(z)$$

(18b)

and

$$G_\rho(k,z) = H_\rho(-k,z)$$

(19)

whence we construct the vertex

$$V_{\rho\mu}(k) = \int d^2z \, G_\rho^\dagger(k,z) H_\mu(k,z)$$

(20)

We note again that, although H_ρ is not strictly covariant, in the combination $G^+ H$ the covariance is regained apart from terms proportional to $(p_{0_a} - p_{0_b})_\mu$ which annihilate all states in the model even those off mass shell.

The spin two part of the vertex (20) has poles at $\alpha(-k^2) = 2, 3, \ldots$ with the residue of the first pole being equal to the conventional spin two vertex[1,7] written in terms of the c operators together with another spin two vertex which decouples from on-mass-shell scalars since it involves the combination (a-b). The trace of vertex (20) has poles at $\alpha(-k^2) = 0, 1, 2, \ldots$ because it contains a piece proportional to the vertex 10. Due to the bilinear nature of the coupling scheme, the construction of a true vector current does not appear to be a straightforward application of this method.

As a second example we consider the choice

$$H_a(k,z) = z^{-k^2/4} \, e^{i\frac{k}{\sqrt{2}} \cdot \tilde{F}_a(z)} \, e^{i\frac{k}{\sqrt{2}} \cdot F_a(z)}$$

(21)

$$G_b(k,z) = H_b(-k,z)$$

(22)

This time both H and G strictly satisfy the requirement of Eq. 2

with $J = -k^2/_4$. The SU(1,1) invariant vertex is then given by

$$V(k) = \int d^2z \, (1-|z|^2)^{k^2/_2 - 2} \, |z|^{-k^2/_2} \, e^{i\frac{k}{\sqrt{2}}[\tilde{F}_a(z) + \tilde{F}_b(\bar{z})]} \, e^{i\frac{k}{\sqrt{2}}[F_a(z) + F_b(\bar{z})]}$$

(23)

and the corresponding N point amplitude is then

$$A_N(k_1, \ldots k_N) = \prod_{i=1}^{N} \left\{ d^2 z_i \, (1-|z_i|^2)^{k_i^2/_2 - 2} \, |z_i|^{-k_i^2/_2} \right\}$$
$$* \prod_{i<j} \left(|z_i - z_j| |\bar{z}_i - \bar{z}_j| \right)^{k_i \cdot k_j/_2}$$

(24)

This expression is equivalent to the one originally proposed by

Drummond[9] to represent the off shell Veneziano amplitude. However,

Eq. 24 has poles in $k_i \cdot k_j$ instead of in $(k_i + k_j)^2$ because z_i can

approach z_j inside the unit circle where there are no compensating

factors from the $1-|z|^2$ terms. Therefore, it is unsuited for an

off shell amplitude. Nevertheless, this amplitude has merits of its

own since it can be regarded as a scattering amplitude for fixed

values of k_i^2 ($\neq 2, -2, -6 \ldots$). In particular at the point

$$\alpha_0 = \frac{k_i^2}{2} = 2 \qquad i = (1, 2, \ldots N)$$

It reduces to the form proposed as the N point generalization of the

Virasoro amplitude[3]. We emphasize that although this amplitude has

higher symmetry for $\alpha_0 = 2$, it has been derived solely from an

SU (1,1) coupling scheme. Indeed the duality of the n -dimensional

generalization of this type of amplitude can be understood in terms of

representations of SU(1,1) in higher dimensional spaces.

Furthermore, for any

$$\alpha_0 = \frac{k_i^2}{2} \neq 1, -1, -3, \cdots$$

this amplitude retains its s-t-u symmetry, and may thus be the

proper continuation in α_0 of the Shapiro-Yoshimura formula.

Still another possible vertex is obtained by choosing

$$H(k, z) = z^{-k^2/8} \, e^{i\frac{k}{2}\cdot \tilde{F}(z)} \, e^{i\frac{k}{2}\cdot F_a(z)}$$

(25a)

$$G(k, z) = H(-k, z)$$

(25b)

Where only the a operators are used, hence we will drop

the a subscripts in what follows. The vertex is

$$V(k) = \int d^2z \, (1-|z|^2)^{k^2/2 - 2} \, |z|^{-k^2/2} \times e^{i\frac{k}{2}\cdot[\tilde{F}(z)+\tilde{F}(\bar{z})]} \, e^{i\frac{k}{2}\cdot[F(z)+F(\bar{z})]}$$

(26)

the corresponding amplitude is

$$A_N = \int \prod_{i=1}^{N} d^2z_i \, (1-|z_i|^2)^{k_i^2/2 - 2} \prod_{i<j} (|z_i-z_j||1-z_i z_j^*|)^{k_i \cdot k_j/2}$$

(27)

which can be shown to have no overlapping poles; however it also

has poles in $k_i k_j$, which make it unsuitable for an off-shell dual

amplitude.

In conclusion we would expect that the scheme presented here,

besides allowing for the above synthesis of recent work, will find

its widest application in conjunction with the introduction of new

oscillators.[10,11,12,13] Thus for example while we have discovered

intrinsic problems with the construction of a vector current in the

conventional Veneziano model, the coupling scheme of Eq. (4)

readily lends itself to the building of a vector current in bilinear

models such as those of Refs. (10) and (11).

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